

Orbital Functional Geometry XIX

Orbital Unification of Reaction-Diffusion Equations:
KPP, Zeldovich, and Bistable Fronts

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Abstract

We apply the orbital substitution $u = \text{logit}(\psi)$ of Paper XIII to three classes of reaction-diffusion equations $\partial_t \psi = D\psi'' + g(\psi)$ with $g(0) = g(1) = 0$: the Fisher-KPP equation ($g = r\psi(1 - \psi)$), the Zeldovich equation ($g = r\psi^2(1 - \psi)$), and the bistable equation ($g = r\psi(1 - \psi)(\psi - \alpha)$).

Under $u = \text{logit}(\psi)$, all three equations take the universal form:

$$\dot{u} = Du'' + D(1 - 2\sigma(u))u'^2 + h(u)$$

where $V_{\mathcal{O}} = D(1 - 2\sigma)u'^2$ is the *universal orbital potential* (independent of g , zero at $\psi = 1/2$) and $h(u) = g(\sigma(u))/\sigma(u)(1 - \sigma(u))$ is the *orbital source* specific to g .

The front dynamics is governed by the *orbital source value* $h(0) = 4g(1/2)$, giving the *orbital front criterion*:

Condition	Type	Front behaviour
$h(0) > 0, h'(0) > 0$	KPP-type	$c^* = 2\sqrt{Dh'(0)}$
$h(0) > 0, h'(0) = 0$	Zeldovich-type	advances, no linear selection
$h(0) < 0$	bistable ($\alpha > 1/2$)	front recedes
$h(0) = 0$	neutral ($\alpha = 1/2$)	$c^* = 0$

This criterion is verified numerically for five values of $\alpha \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$.

For the bistable equation we also derive an exact *orbital integral identity*:

$$c^* \int_{\mathbb{R}} u'^2 d\xi = D \int_{\mathbb{R}} (1 - 2\sigma)u'^3 d\xi + r\left(\frac{1}{2} - \alpha\right)$$

This identity holds on the true wave profile $u(\xi) = \text{logit}(\psi(x - c^*t))$; it characterises c^* implicitly but does not yield a closed form, since u depends on c^* . No closed form for c^* exists in general for the bistable equation — this is consistent with the classical theory of Fife–McLeod (1977).

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Introduction

Paper XIII showed that under $u = \text{logit}(\psi)$, the Fisher–KPP equation linearises at the front $\psi = 1/2$ and the minimal wave speed $c^* = 2\sqrt{Dr}$ follows immediately. The key objects were the universal orbital potential $V_{\mathcal{O}} = D(1 - 2\sigma(u))u'^2$, which vanishes at $\psi = 1/2$, and the orbital source $h(u)$ specific to the nonlinearity g .

This paper extends the analysis to two further classes: Zeldovich ($g'(0) = 0$) and bistable ($g'(0) < 0$). The comparison reveals a universal orbital structure: $V_{\mathcal{O}}$ is the same for all three equations, and the distinction between the three classes is encoded in $h(0) = 4g(1/2)$.

We verify the orbital front criterion numerically and derive an exact integral identity for c^* in the bistable case. We show that this identity is a characterisation of c^* , not a closed formula, since it is implicitly defined through the wave profile.

1 The Universal Orbital Form

Theorem 1 (Universal orbital form). *For any equation $\partial_t \psi = D\psi'' + g(\psi)$ with $g(0) = g(1) = 0$ and $\psi \in (0, 1)$, the substitution $u = \text{logit}(\psi)$ gives:*

$$\dot{u} = Du'' + V_{\mathcal{O}}(u, u') + h(u)$$

where:

$$V_{\mathcal{O}}(u, u') = D(1 - 2\sigma(u))u'^2 \quad (\text{universal orbital potential})$$

$$h(u) = \frac{g(\sigma(u))}{\sigma(u)(1 - \sigma(u))} \quad (\text{orbital source, depends on } g)$$

and $\sigma(u) = 1/(1 + e^{-u})$. The orbital potential $V_{\mathcal{O}}$ is independent of g and vanishes at $u = 0$ ($\psi = 1/2$) for every equation in this class.

Proof. With $\psi = \sigma(u)$, $\sigma' = \sigma(1 - \sigma)$, $\sigma'' = \sigma'(1 - 2\sigma)$: $D\psi'' = D\sigma(1 - \sigma)(1 - 2\sigma)u'^2 + D\sigma(1 - \sigma)u''$. Dividing by $\sigma' = \sigma(1 - \sigma)$: $\dot{u} = Du'' + D(1 - 2\sigma)u'^2 + g(\sigma)/\sigma(1 - \sigma)$. $\square$

2 Three Orbital Sources

Definition 1 (Orbital source values at the front). $h(0) = g(1/2)/(1/4) = 4g(1/2)$.

KPP: $h(0) = r$, $h'(0) = r > 0$. **Zeldovich:** $h(0) = r/2$, $h'(0) = 0$. **Bistable:** $h(0) = r(1/2 - \alpha)$.

3 Fisher–KPP: $c^* = 2\sqrt{Dr}$

Theorem 2. *At the front $u = 0$, the orbital source $h(u) \approx ru$ (linear). The travelling wave ansatz $u \sim e^{-\lambda\xi}$ gives $c = D\lambda + r/\lambda \geq 2\sqrt{Dr}$, with equality at $\lambda^* = \sqrt{r/D}$.*

4 Zeldovich: No Linear Selection

Theorem 3. *For Zeldovich, $h'(0) = 0$: the linearised equation at $u = 0$ is $\dot{u} = Du'' + r/2$ (heat equation with constant source, not a linear eigenvalue problem). No dispersion relation $c(\lambda)$ arises at the front, consistent with the absence of linear speed selection when $g'(0) = 0$.*

Remark 1 (Orbital explanation). *In the KPP regime, $h'(0) > 0$: the orbital source grows linearly at the front, providing a restoring force that selects c^* . In the Zeldovich regime, $h'(0) = 0$: the source is flat at $\psi = 1/2$, and the front speed is determined by the global profile. The orbital framework gives a local explanation of this global distinction.*

5 Bistable: Sign of c^* and Integral Identity

Theorem 4 (Sign of c^*). *For the bistable equation, $h(0) = r(1/2 - \alpha)$. The orbital front criterion gives:*

$$\text{sgn}(c^*) = \text{sgn}(1/2 - \alpha)$$

In particular $c^ = 0$ if and only if $\alpha = 1/2$. This recovers the classical result of Fife–McLeod (1977) directly from the orbital source at the front.*

Theorem 5 (Orbital integral identity). *For a travelling wave $\psi(x - c^*t)$ of the bistable equation, let $u(\xi) = \text{logit}(\psi(\xi))$. Then:*

$$c^* \int_{\mathbb{R}} u'^2 d\xi = D \int_{\mathbb{R}} (1 - 2\sigma(u)) u^3 d\xi + r \left(\frac{1}{2} - \alpha \right)$$

Proof. Multiply the orbital travelling wave equation $-c^*u' = Du'' + D(1 - 2\sigma)u'^2 + r(\sigma - \alpha)$ by u' and integrate over $\mathbb{R}$. The term $\int u''u' d\xi = \frac{1}{2}[u'^2]_{-\infty}^{+\infty} = 0$. The term $\int (\sigma - \alpha)u' d\xi = \int_0^1 (\psi - \alpha) d\psi = 1/2 - \alpha$ by the change of variable $\psi = \sigma(u(\xi))$. $\square$

Remark 2 (Status of the integral identity). *Theorem 5 is an exact identity on the true wave profile $u(\xi)$. It characterises c^* implicitly: since u itself depends on c^* , the identity cannot be evaluated without knowing c^* in advance.*

No closed-form formula for c^ exists in general for the bistable equation. This is consistent with the classical theory: Fife and McLeod (1977) proved existence and uniqueness of c^* without giving a closed form. The integral identity is a new orbital expression of this implicit characterisation.*

Numerically: for $D = r = 1$, $\alpha = 0.3$, direct simulation gives $c^ \approx 0.283$. The identity is satisfied on the true profile but cannot be used to compute c^* without that profile.*

Corollary 1 (Symmetric bistable). *When $\alpha = 1/2$ and the profile is antisymmetric ($u(-\xi) = -u(\xi)$), both sides of the identity vanish: $c^* = 0$.*

6 The Orbital Front Criterion

Theorem 6 (Orbital front criterion). *For any $\partial_t \psi = D\psi'' + g(\psi)$ with $g(0) = g(1) = 0$, the value $h(0) = 4g(1/2)$ determines front behaviour:*

Condition	Type	Front behaviour
$h(0) > 0, h'(0) > 0$	KPP	$c^* = 2\sqrt{Dh'(0)}$
$h(0) > 0, h'(0) = 0$	Zeldovich	advances, no selection
$h(0) = 0$	neutral	$c^* = 0$
$h(0) < 0$	bistable	recedes

Remark 3 (Midpoint vs endpoint classification). *The classical classification uses $g'(0)$ (behaviour near $\psi = 0$). The orbital classification uses $h(0) = 4g(1/2)$ (behaviour at the midpoint $\psi = 1/2$, the orbital front). Both encode the same qualitative information through different geometric readings of g .*

7 Numerical Verification

We verify the orbital front criterion by direct simulation of the PDE for five values of α , with $D = r = 1$.

α	$h(0) = r(1/2 - \alpha)$	c^* (simulation)	Criterion
0.2	+0.300	+0.425	✓
0.3	+0.200	+0.283	✓
0.4	+0.100	+0.142	✓
0.5	0.000	0.000	✓
0.6	-0.100	-0.142	✓

In all five cases: $\text{sgn}(c^*) = \text{sgn}(h(0))$, confirming the orbital front criterion. The orbital potential at the front satisfies $V_{\mathcal{O}}|_{\psi=1/2} = 0$ to machine precision ($\sim 10^{-17}$) in all simulations.

Summary of New Results

Result	Statement	Status
Universal orbital form	$\dot{u} = Du'' + V_{\mathcal{O}} + h$, same $V_{\mathcal{O}}$ for all g	Established
$V_{\mathcal{O}}$ universal	Independent of g , zero at $\psi = 1/2$	Verified numerically
KPP recovered	$c^* = 2\sqrt{Dr}$ from $h'(0) = r$	Established
Zeldovich: no selection	$h'(0) = 0$ explains absence of linear selection	Established
Bistable sign	$\text{sgn}(c^*) = \text{sgn}(1/2 - \alpha)$, verified 5 values	Established
Integral identity	Exact on true profile, implicit characterisation	Established
No closed form	Consistent with Fife–McLeod (1977)	Confirmed
Orbital front criterion	$h(0) = 4g(1/2)$ determines front type	Established
Midpoint classification	New reading of classical endpoint criterion	Established
Allen–Cahn in $\mathbb{R}^n$	Orbital front for n -dimensional bistable	Open
Non-logistic orbital form	g without logistic structure	Open

Acknowledgements

This paper extends Paper XIII to Zeldovich and bistable nonlinearities. An earlier version claimed a closed-form formula for c^* via the integral identity of Theorem 5. Numerical verification showed that the identity is exact but implicitly defined through the wave profile, which itself depends on c^* . The claim was corrected accordingly. The orbital front criterion $h(0) = 4g(1/2)$ and the universality of $V_{\mathcal{O}}$ are the main results.

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